# Online Appendix A Theory of Falling Growth and Rising Rents 

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## A Stylized facts

Fact 1: Slow growth interrupted by a burst of growth. Figure 1 in the paper, reproduced here as Figure Ala, presents U.S. annual TFP growth in Manufacturing, Trade and Service industries from the Bureau of Labor Statistics (BLS) KLEMS data. The Figure shows growth accelerating from its 1988-1995 average of $0.8 \%$ per year to $2.1 \%$ per year 1996-2005, before falling to just $0.4 \%$ per year 2006-2019. Figure A1b shows that in all non-farm private industries, U.S. annual TFP growth accelerated from its 1948-1995 average of $1.81 \%$ per year to $2.86 \%$ per year from 1996 -2005, before falling to $1.16 \%$ per year from 2006-2018.

Figure A1: Productivity growth


Source: (a) BLS KLEMS multifactor productivity series. We calculate yearly productivity growth in two digit NAICS manufacturing, trade and service industries by adding R\&D and IP contribution to BLS MFP and then expressing the sum in labor augmenting form. We aggregating industry growth rates using industry share of labor costs. (b) BLS multifactor productivity series. We calculate yearly productivity growth rate by adding R\&D and IP contribution to BLS MFP and then converting the sum to labor augmenting form. Both figures plot the average productivity growth within each subperiod. The unit is percentage points.

Fact 2: Rising concentration. Table Al presents the average change from 1982 to 2012 in top 20 firm concentration within 4-digit NAICS inside Manufacturing, Retail Trade, Wholesale Trade, and Service industries,
respectively. ${ }^{1}$ These results are from firm-level data in U.S. Census years. Top 20 concentration rose in all sectors.

Table A1: Cumulative change in concentration 1982-2012 (ppt)

|  | MFG | RET | WHO | SRV |
| :---: | :---: | :---: | :---: | :---: |
| Top 20 firms sales share 1982 | 70 | 29 | 45 | 21 |
| Top 20 firms sales share 2012 | 74 | 46 | 57 | 27 |
| Change | 4 | 17 | 12 | 6 |

Source: Figure IV of Autor et al. (2020). Concentration in each industry are averages across 4 -digit industries, with the industries weighted by industry sales shares.

Table A2 displays the ratio of sales to employment of the top 20 firms relative to smaller firms in the four sectors. The ratio increased in manufacturing but has been stable in other sectors.

Table A2: Sales/employment of top 20 firms relative to remaining firms

|  | MFG | RET |  | WHO |  | SRV |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 1982-1992 average | 1.28 |  | 1.19 |  | 2.76 |  |

Source: Figure IV of Autor et al. (2020).

Figure A2a shows the number of establishments per firm from 1980 to 2014 in three size bins based on U.S. Census Bureau Business Dynamic Statistics. Firms with 10,000+ employees added establishments steadily starting in the early 1990s, when TFP growth accelerated. ${ }^{2}$

[^0]Figure A2b shows the rate at which large firms added new establishments relative to their stock of establishments. This rate can be viewed as a crude proxy for their pace of product innovation. The largest firms experienced a burst of establishment entry in the 1990s, which receded from 2005 onward. Again, these broad trends dovetail with the acceleration and deceleration of TFP growth.

Figure A2: Establishments per firm by firm size in manufacturing, trade and service industries


Source: U.S. Census Bureau Business Dynamic Statistics. Left-hand side panel plots the number of establishments per firm relative to 1990 within employment bins. Righthand side panel shows the number of new establishments over the total number of establishments for different firm size bins. The lines represent 10-year centered moving average, relative to 1990 .

## Fact 3: Reallocation of market share toward low labor share firms.

 According to Autor et al. (2020), within Manufacturing, Trade and Services sectors, sales were reallocated to low labor share firms. Table A3 reproduces statistics from Autor et al. (2020) showing that the "between" firm component pushed labor share down from 1982-2012 in each of these sectors. Within-firm labor shares actually rose in trade and services. While within-firm labor share declined for manufacturing, the decline is small relative to the decline via the between component.A complementary fact which Autor et al. (2020) document is that larger firms tend to have lower labor shares. Within four-digit industries, the

Table A3: Cumulative change in labor share 1982-2012 (ppt)

| $\Delta \frac{\text { Payroll }}{\text { Sales }}$ | MFG | RET | WHO | SRV |
| :---: | :---: | :---: | :---: | :---: |
|  | -6.73 | -0.86 | -0.08 | 0.24 |
| Within firm | -1.71 | 4.39 | 4.66 | 1.73 |
| Between | -4.54 | -5.44 | -4.59 | -0.76 |

Source: Figure VIII in Autor et al. (2020). This is a Melitz-Polanec (Melitz and Polanec, 2015) decomposition of the change in the labor share. The entry and exit margin is not reported. The unit is percentage points. MFG, RET, WHO, and SRV stand for manufacturing, retail, wholesale, and service.
elasticity of firm labor share with respect to firm sales averages -3.1 across these four Census sectors.

## B Proofs and derivations

## B. 1 Solution of the firm's problem

This section derives the solution to the firm problem when the economy is on the BGP. In this case the firm profit functions $\pi_{H}$ and $\pi_{L}$ are quadratic functions of $n$, the policy and value functions can be characterized in closed form. We state the solution in the next proposition.

Proposition A1 Let us denote $\widetilde{\pi}_{H}\left(S^{\star}\right) \equiv S^{\star}\left(1-\frac{1}{\gamma}\right)+\left(1-S^{\star}\right)\left(1-\frac{1}{\Delta \gamma}\right)$ and $\widetilde{\pi}_{L}\left(S^{\star}\right) \equiv$ $S^{\star}\left(1-\frac{\Delta}{\gamma}\right)+\left(1-S^{\star}\right)\left(1-\frac{1}{\gamma}\right)$. Also, we define

$$
\bar{n}_{k}\left(S^{\star}, z^{\star}\right) \equiv \frac{\widetilde{\pi}_{k}\left(S^{\star}\right)+\left(1-z^{\star}\right) \psi_{r}-\frac{\psi_{r}}{\beta}}{\psi_{o}}
$$

for $k=H, L$. For a given $S^{\star} \in(0,1)$ and $z^{\star} \in(0,1)$, the policy functions $f_{k}(n)$, $k=H, L$ are given by

$$
f_{k}(n)= \begin{cases}\left(1-z^{\star}\right) n & \text { if } n \geq \frac{\bar{n}_{k}\left(S^{\star}, z^{\star}\right)}{1-z^{\star}}  \tag{A1}\\ \bar{n}_{k}\left(S^{\star}, z^{\star}\right) & \text { otherwise. }\end{cases}
$$

Let $m$ denote the smallest integer such that $n<\frac{\bar{n}_{k}\left(S^{\star}, z^{\star}\right)}{\left(1-z^{\star}\right)^{m+1}}$ and $\bar{n}_{k}$ be a shorthand for $\bar{n}_{k}\left(S^{\star}, z^{\star}\right)$. The value functions are given by

$$
v_{k}(n)= \begin{cases}\left(\widetilde{\pi}_{k} \bar{n}_{k}-\frac{1}{2} \psi_{o} \bar{n}_{k}^{2}-\psi_{r} z^{\star} \bar{n}_{k}\right) \frac{1}{1-\beta} & \text { if } n=\bar{n}_{k}  \tag{A2}\\ \widetilde{\pi}_{k} n-\frac{1}{2} \psi_{o} n^{2}-\psi_{r}\left(\bar{n}_{k}-\left(1-z^{\star}\right) n\right)+\beta v_{k}\left(\bar{n}_{k}\right) & \text { if } n<\frac{\bar{n}_{k}}{\left(1-z^{\star}\right)} \\ \widetilde{\pi}_{k} n \frac{1-\left(\beta\left(1-z^{\star}\right)\right)^{m+1}}{1-\beta\left(1-z^{\star}\right)}-\frac{1}{2} \psi_{o} n^{2} \frac{1-\left(\beta\left(1-z^{\star}\right)^{2}\right)^{m+1}}{1-\beta\left(1-z^{\star}\right)^{2}} & \text { Otherwise } \\ +\psi_{r} n \beta^{m}\left(1-z^{\star}\right)^{m+1}+\beta^{m+1} v_{k}\left(\bar{n}_{k}\right)-\beta^{m} \psi_{r} \bar{n}_{k} & \end{cases}
$$

for $k=H, L$.

Proposition Al says that the $\bar{n}_{k}\left(S^{\star}, z^{\star}\right)$ is the optimal level of $n$ for a firm of type $k=H, L$ on the BGP. A firm invests just enough to hit $\bar{n}_{k}$ in the next period. If however $n$ is too high such that even without investing (and letting $n$ decay at
rate $\left.z^{\star}\right) \bar{n}_{k}$ is not reached next period, the non-negativity constraint on $\mathrm{R} \& \mathrm{D}$ is binding and the firm invests zero for $m$ periods.

## B. 2 Comparative statics on equilibrium BGP

In this Section, we establish the comparative statics results in Table 1 that summarizes how the equilibrium values $S^{\star}, \lambda_{H}^{\star}, \lambda_{L}^{\star}, \lambda^{\star}, z^{\star}$, and $g^{\star}$ respond to changes in the parameters $\psi_{o}, \Delta, \gamma$, and $\psi_{r}$.

In all the following, we assume that the conditions (14) and (15) hold, such that an interior balanced growth path exists (see Proposition 2).

## Equilibrium concentration $S^{\star}$

From the paper, equation (25) shows that

$$
S^{\star}=\frac{\phi+\frac{(\Delta-1) J \phi(1-\phi)}{\gamma \Delta \psi_{o}}}{1-\frac{(\Delta-1)^{2} J \phi(1-\phi)}{\gamma \Delta \psi_{o}}}
$$

Taking derivatives we get

$$
\frac{\partial S^{\star}}{\partial \psi_{o}}=-\frac{[1+\phi(\Delta-1)] \frac{\Delta-1}{\Delta} \frac{J}{\gamma \psi_{o}^{2}(1-\phi) \phi}}{\left(\frac{1}{(1-\phi) \phi}-\frac{(\Delta-1)^{2}}{\gamma \Delta} \frac{J}{\psi_{o}}\right)^{2}}<0,
$$

and

$$
\frac{\partial S^{\star}}{\partial \gamma}=-\frac{[1+\phi(\Delta-1)] \frac{\Delta-1}{\Delta} \frac{J}{\gamma^{2} \psi_{o}(1-\phi) \phi}}{\left(\frac{1}{(1-\phi) \phi}-\frac{(\Delta-1)^{2}}{\gamma \Delta} \frac{J}{\psi_{o}}\right)^{2}}<0
$$

We also directly see that $\frac{\partial S^{\star}}{\partial \psi_{r}}=0$. Finally, using the fact that $\frac{\Delta-1}{\Delta}$ and $\frac{(\Delta-1)^{2}}{\Delta}$ are both increasing functions of $\Delta$, we also see that $\frac{\partial S^{\star}}{\partial \Delta}>0$. More precisely, we have

$$
\frac{\partial S^{\star}}{\partial \Delta}=\frac{S^{\star}-\phi}{\Delta(\Delta-1)(1+\phi(\Delta-1))}\left(1+S^{\star}\left(\Delta^{2}-1\right)\right)>0
$$

because $S^{\star}>\phi$ along an interior balanced growth path.

## Equilibrium labor share in high-productivity firms $\lambda_{H}^{\star}$

From (27) in the main text we have

$$
\lambda_{H}^{\star}=S^{\star} \frac{1}{\gamma}+\left(1-S^{\star}\right) \frac{1}{\gamma \Delta}=\frac{S^{\star}}{\gamma} \frac{\Delta-1}{\Delta}+\frac{1}{\gamma \Delta} .
$$

Both $\psi_{o}$ and $\psi_{r}$ potentially only affect $\lambda_{H}^{\star}$ through their effects on $S^{\star}$. As $\lambda_{H}^{\star}$ is monotonically increasing in $S^{\star}, \psi_{o}$ and $\psi_{r}$ affect $\lambda_{H}^{\star}$ in exactly the same way as they affect $S^{\star}$. Hence, we have

$$
\frac{\partial \lambda_{H}^{\star}}{\partial \psi_{o}}<0 \text { and } \frac{\partial \lambda_{H}^{\star}}{\partial \psi_{r}}=0 .
$$

An increases in $\gamma$ decreases $S^{\star}$ and adds an additional negative effect on $\lambda_{H}^{\star}$ as $\gamma$ directly shows up in the denominator of (27). We therefore have unambiguously $\partial \lambda_{H}^{\star} / \partial \gamma<0$.

Finally, we have

$$
\frac{\partial \lambda_{H}^{\star}}{\partial \Delta}=\frac{1}{\gamma}\left[\frac{\partial S^{\star}}{\partial \Delta}\left(1-\frac{1}{\Delta}\right)-\frac{1-S^{\star}}{\Delta^{2}}\right]
$$

which is positive if and only if the following condition (A3) is satisfied:

$$
\begin{equation*}
\frac{\psi_{o} \gamma}{J}<(2+\phi(\Delta-1)) \phi(\Delta-1) \tag{A3}
\end{equation*}
$$

## Equilibrium labor share in low-productivity firms $\lambda_{L}^{\star}$

Equation (28) states

$$
\lambda_{L}^{\star}=S^{\star} \frac{\Delta}{\gamma}+\left(1-S^{\star}\right) \frac{1}{\gamma}=\frac{S^{\star}}{\gamma}(\Delta-1)+\frac{1}{\gamma} .
$$

As we have $\lambda_{L}^{\star}=\Delta \lambda_{H}^{\star}$ the signs of the comparative static effects of $\psi_{o}, \psi_{r}$ and $\gamma$ on $\lambda_{L}^{\star}$ are the same as the ones on $\lambda_{H}^{\star}$.

The effect from $\Delta$ on $\lambda_{L}^{\star}$ is given by

$$
\frac{\partial \lambda_{L}^{\star}}{\partial \Delta}=\frac{1}{\gamma}\left[\frac{\partial S^{\star}}{\partial \Delta}(\Delta-1)+S^{\star}\right]>0
$$

which is strictly positive as $\partial S^{\star} / \partial \Delta>0$.

## Equilibrium labor share $\lambda^{\star}$

We have $\lambda^{\star}=\lambda_{L}^{\star}\left(1-S^{\star}\right)+\lambda_{H}^{\star} S^{\star}$ and $\lambda_{L}^{\star}=\Delta \lambda_{H}^{\star}$, which imply

$$
\lambda^{\star}=\lambda_{L}^{\star}\left(1-S^{\star} \frac{\Delta-1}{\Delta}\right)=\frac{1}{\gamma}\left(1+S^{\star}(\Delta-1)\right)\left(1-S^{\star} \frac{\Delta-1}{\Delta}\right) .
$$

As $S^{\star}$ is independent of $\psi_{r}$ this shows that $\lambda^{\star}$ is not impacted by $\psi_{r}$. For the effect of $\psi_{o}$, we have

$$
\frac{\partial \lambda^{\star}}{\partial S^{\star}}=\lambda_{H}^{\star}+S^{\star} \frac{\Delta-1}{\gamma \Delta}-\lambda_{L}^{\star}+\left(1-S^{\star}\right) \frac{\Delta-1}{\gamma}=\frac{\partial \lambda^{\star}}{\partial S^{\star}}=\frac{(\Delta-1)^{2}}{\gamma \Delta}\left(1-2 S^{\star}\right),
$$

which leads to

$$
\frac{\partial \lambda^{\star}}{\partial \psi_{o}}=\frac{\partial \lambda^{\star}}{\partial S^{\star}} \frac{\partial S^{\star}}{\partial \psi_{o}}=\frac{(\Delta-1)^{2}}{\gamma \Delta}\left(1-2 S^{\star}\right) \frac{\partial S^{\star}}{\partial \psi_{o}} .
$$

Since $S^{\star}$ is decreasing in $\psi_{o}$ this implies that the aggregate labor income share increases in $\psi_{o}$ if and only if $S^{\star}>1 / 2$. In terms of exogenous parameters this condition reads

$$
\begin{equation*}
\frac{\gamma \psi_{o}}{J}<\frac{\left(\Delta^{2}-1\right) \phi(1-\phi)}{\Delta(1-2 \phi)} \tag{A4}
\end{equation*}
$$

The impact of $\gamma$ can be established by evaluating

$$
\frac{\partial \lambda^{\star}}{\partial \gamma}=-\frac{\lambda^{\star}}{\gamma}+\frac{1}{\gamma} \frac{\partial S^{\star}}{\partial \gamma}\left(1-2 S^{\star}\right) \frac{(\Delta-1)^{2}}{\Delta}
$$

Noting that

$$
\frac{\partial S^{\star}}{\partial \gamma}=-\frac{S^{\star}-\phi}{\gamma} \frac{1+S^{\star}(\Delta-1)}{1+\phi(\Delta-1)}
$$

we finally have

$$
\frac{\partial \lambda^{\star}}{\partial \gamma}=-\frac{1+S^{\star}(\Delta-1)}{\gamma^{2}}\left[1-S^{\star} \frac{\Delta-1}{\Delta}+\frac{\left(S^{\star}-\phi\right)\left(1-2 S^{\star}\right)}{1+\phi(\Delta-1)} \frac{(\Delta-1)^{2}}{\Delta}\right]
$$

So $\frac{\partial \lambda^{\star}}{\partial \gamma}<0$ if and only if

$$
\begin{equation*}
1-S^{\star} \frac{\Delta-1}{\Delta}+\frac{\left(S^{\star}-\phi\right)\left(1-2 S^{\star}\right)}{1+\phi(\Delta-1)} \frac{(\Delta-1)^{2}}{\Delta}>0 \tag{A5}
\end{equation*}
$$

Regarding the impact of $\Delta$, we have

$$
\frac{\partial \lambda^{\star}}{\partial \Delta}=\frac{1}{\gamma \Delta} \frac{\partial S^{\star}}{\partial \Delta}(\Delta-1)^{2}\left(1-2 S^{\star}\right)+\frac{\Delta^{2}-1}{\gamma \Delta^{2}} S^{\star}\left(1-S^{\star}\right)
$$

which is positive if and only if

$$
\begin{equation*}
\frac{\left(S^{\star}-\phi\right)(\Delta-1)}{1+(\Delta-1) S^{\star}}\left(1+S^{\star}\left(\Delta^{2}-1\right)\right)\left(1-2 S^{\star}\right)+\left(\Delta^{2}-1\right) S^{\star}\left(1-S^{\star}\right)>0 . \tag{A6}
\end{equation*}
$$

There is no simple expression of conditions (A5) and (A6) in terms of exogenous parameters. However, these conditions are satisfied for parameters that generate $S^{\star} \leq 1 / 2$. Therefore, the converse of (A4) or

$$
\frac{\gamma \psi_{o}}{J} \geq \frac{\left(\Delta^{2}-1\right) \phi(1-\phi)}{\Delta(1-2 \phi)}
$$

is a sufficient condition for labor share to decrease with $\gamma$ and to increase with $\Delta$.

## Equilibrium rate of creative destruction $z^{\star}$

Equation (26) in the paper shows that

$$
z^{\star}=\frac{1}{\psi_{r}}-\frac{1-\beta}{\beta}-\frac{1}{\psi_{r}} \frac{\frac{\psi_{o}}{J}+\frac{1}{\gamma}}{1-(1-\phi) \phi \frac{(\Delta-1)^{2}}{\gamma \Delta} \frac{J}{\psi_{o}}}
$$

Taking derivatives of this with respect to $\psi_{o}$ gives

$$
\frac{\partial z^{\star}}{\partial \psi_{o}}=\frac{2 \phi(1-\phi) \frac{(\Delta-1)^{2}}{\gamma \Delta \psi_{o}}+\phi(1-\phi) \frac{(\Delta-1)^{2}}{\gamma \Delta \psi_{o}} \frac{J}{\gamma \psi_{o}}-\frac{1}{J}}{\psi_{r}\left(1-(1-\phi) \phi \frac{(\Delta-1)^{2}}{\gamma \Delta} \frac{J}{\psi_{o}}\right)^{2}}
$$

This expression is positive if and only if

$$
\begin{equation*}
\frac{J(\Delta-1)^{2}}{\gamma \Delta \psi_{o}}\left(\frac{J}{\gamma \psi_{o}}+2\right)>\frac{1}{\phi(1-\phi)} \tag{A7}
\end{equation*}
$$

We also immediately see from (26) that $\frac{\partial z^{\star}}{\partial \psi_{r}}<0$ (recall that $z^{\star}>0$, so that the term $\frac{1}{\psi_{r}}$ multiplies a positive number) and that $\frac{\partial z^{\star}}{\partial \gamma}>0$ (note that (14) ensures that $\left.1>(1-\phi) \phi \frac{(\Delta-1)^{2}}{\gamma \Delta} \frac{J}{\psi_{o}}\right)$.

Finally, $\frac{\partial z^{\star}}{\partial \Delta}<0$ results from the fact that $\frac{(\Delta-1)^{2}}{\Delta}$ is an increasing function of $\Delta>1$.

## Equilibrium growth rate $g^{\star}$

As $g^{\star}=\gamma^{z^{\star}}-1$, the effects of increases in $\Delta, \psi_{o}$ and $\psi_{r}$ are the same are the effects of such increases on $z^{\star}$. Finally, given that $z^{\star}$ increases with $\gamma$, then the impact of $\gamma$ on $g^{\star}$ is even more positive.

## C Extensions

## C. 1 CRRA preferences and CES production structure

The Cobb-Douglas model is tractable and provides analytical characterization. In this section we lay out a general model with a CES production structure and CRRA preferences. With a constant elasticity of substitution $\sigma>1$, products with higher quality or productivity have higher market shares. Also, the price setting of a high productivity firm facing a low productivity second-best firm may no longer be constrained and such a firm instead may simply charge the monopoly markup $\frac{\sigma}{\sigma-1}$.

## CRRA preferences

Instead of log preferences we generalize utility to the CRRA class

$$
U_{0}=\sum_{t=0}^{\infty} \beta^{t} \frac{C_{t}^{1-\theta}-1}{1-\theta}
$$

Then, the resulting Euler equation from household's optimization is given by

$$
\begin{equation*}
\frac{C_{t+1}}{C_{t}}=\left[\beta\left(1+r_{t+1}\right)\right]^{\frac{1}{\theta}} \tag{A8}
\end{equation*}
$$

Hence, the relationship between net growth rate $g^{\star}$ and the interest rate on the BGP is given by $1+g^{\star}=\left[\beta\left(1+r^{\star}\right)\right]^{\frac{1}{\theta}}$.

## CES production

Instead of the Cobb-Douglas technology we assume a more general CES technology in final production

$$
\begin{equation*}
Y=\left(\int_{0}^{1}[q(i) y(i)]^{\frac{\sigma-1}{\sigma}} d i\right)^{\frac{\sigma}{\sigma-1}} \tag{A9}
\end{equation*}
$$

Here $y(i)$ denotes the quantity and $q(i)$ the quality of product $i$. This new structure yields as demand for product $i$

$$
\begin{equation*}
y(i)=q(i)^{\sigma-1}\left(\frac{P}{p(i)}\right)^{\sigma} Y \tag{A10}
\end{equation*}
$$

where we have the (new) aggregate price index given by

$$
\begin{equation*}
P=\left(\int_{0}^{1}[p(i) / q(i)]^{1-\sigma} d i\right)^{\frac{1}{1-\sigma}} \tag{Al1}
\end{equation*}
$$

We normalize this aggregate price index again to one in each period.

## Solving for the BGP in this more general model

The rest of the model is unchanged. In particular we still have two process efficiency types and the productivity differential is captured by $\Delta$. We now solve for the BGP in this model.

Together with the definition of the numéraire the demand (A10) gives for per-period profit in a line

$$
\begin{equation*}
Y P\left(\frac{P}{p(i) / q(i)}\right)^{\sigma-1}\left(1-\frac{1}{\mu(i)}\right) \tag{Al2}
\end{equation*}
$$

With $\sigma>1$ (which is the empirically relevant case we will focus on) there is an optimal markup factor of $\frac{\sigma}{\sigma-1}$. So depending whether the marginal cost of the second-best firm are binding or not we have the following three cases of markups in a line $i$ :

1. In the case of a high type $(\mathrm{H})$ facing a low type $(\mathrm{L})$ second-best firm

$$
\begin{equation*}
\mu_{H L}=\min \left\{\gamma \Delta, \frac{\sigma}{\sigma-1}\right\} \tag{Al3}
\end{equation*}
$$

2. In the case that both leader and second-best firm are of the same type

$$
\begin{equation*}
\mu_{H H}=\mu_{L L}=\min \left\{\gamma, \frac{\sigma}{\sigma-1}\right\} \tag{A14}
\end{equation*}
$$

3. In the case that a low type (L) facing a high type (H) second-best firm

$$
\begin{equation*}
\mu_{L H}=\min \left\{\frac{\gamma}{\Delta}, \frac{\sigma}{\sigma-1}\right\} . \tag{A15}
\end{equation*}
$$

With the CES structure the demand for a product line $i$ depends also on the particular quality of this line (relative to the other lines). But because there is no possibility to target the innovation activity to particular lines and all firms draw repetitively in BGP from the same distribution, the quality level in line $i$ is uncorrelated with the identity of the leading or second-best firm (and therefore uncorrelated with the markup). Since the law of large number applies, on the BGP, each firm has in a given period $t$ the same distribution of quality levels across the different lines.

In the following let us define the "average quality" by

$$
\begin{equation*}
Q_{t}=\left(\int_{0}^{1}\left[q_{t}(i)\right]^{\sigma-1} d i\right)^{\frac{1}{\sigma-1}} \tag{A16}
\end{equation*}
$$

Since the quality of a line is independent of its markup we can write the aggregate price index, (A11), as $P=1=\frac{\widetilde{P}_{t}}{Q_{t}}$, where

$$
\widetilde{P}_{t}=w_{t}\left(\int_{0}^{1}[\mu(i) / \varphi(j(i))]^{1-\sigma} d i\right)^{\frac{1}{1-\sigma}} .
$$

On the BGP we have

$$
\widetilde{P}_{t}=\frac{w_{t}}{\varphi_{L}}\left[\left(S^{\star}\right)^{2}\left(\frac{\mu_{H H}}{\Delta}\right)^{1-\sigma}+S^{\star}\left(1-S^{\star}\right)\left(\frac{\mu_{H L}}{\Delta}\right)^{1-\sigma}+S^{\star}\left(1-S^{\star}\right) \mu_{L H}^{1-\sigma}+\left(1-S^{\star}\right)^{2} \mu_{L L}^{1-\sigma}\right]^{\frac{1}{1-\sigma}} .
$$

The profit in a given line is given by (A12). The sum of profits (before overhead cost) of a high type firm that is active in $n(j)$ lines and is facing in a fraction $S^{\star}$
of them a high second-best firm is given by ${ }^{3}$

$$
n(j) Y\left[S^{\star} \frac{\widetilde{P}_{t}^{\sigma-1}}{\left(\frac{\mu_{H H} w_{t}}{\varphi_{L} \Delta}\right)^{\sigma-1}}\left(1-\frac{1}{\mu_{H H}}\right)+\left(1-S^{\star}\right) \frac{\widetilde{P}_{t}^{\sigma-1}}{\left(\frac{\mu_{H L} w_{t}}{\varphi_{L} \Delta}\right)^{\sigma-1}}\left(1-\frac{1}{\mu_{H L}}\right)\right]=n(j) Y \cdot \widetilde{\pi}_{H}
$$

where we define $\widetilde{\pi}_{H}$ to be equal to the term squared brackets. Similarly, the sum of profits before overhead of a L-type firm having $n(j)$ lines and facing in a fraction $S^{\star}$ of them a high second-best firm is

$$
n(j) Y\left[S^{\star} \frac{\widetilde{P}_{t}^{\sigma-1}}{\left(\frac{\mu_{L H} w_{t}}{\varphi_{L}}\right)^{\sigma-1}}\left(1-\frac{1}{\mu_{L H}}\right)+\left(1-S^{\star}\right) \frac{\widetilde{P}_{t}^{\sigma-1}}{\left(\frac{\mu_{L L} w_{t}}{\varphi_{L}}\right)^{\sigma-1}}\left(1-\frac{1}{\mu_{L L}}\right)\right]=n(j) Y \cdot \widetilde{\pi}_{L}
$$

where we again define $\widetilde{\pi}_{L}$ accordingly. Hence, the new firm profit functions of H and L types relative to GDP, $Y$, become on the BGP:

$$
\begin{aligned}
& \pi_{H}(n)=n \widetilde{\pi}_{H}-\frac{1}{2} \psi_{o} n^{2} \\
& \pi_{L}(n)=n \widetilde{\pi}_{L}-\frac{1}{2} \psi_{o} n^{2}
\end{aligned}
$$

A BGP is characterized as before just with these new profit functions and a different relationship between $\beta, r^{\star}$ and $1+g^{\star}=\frac{Q_{t}}{Q_{t-1}}$ (as specified in the Euler equation).

## BGP characterization

Let us again denote the value of a firm $V$ relative to total output $Y$ by $v \equiv V / Y$. On the BGP (with $h(j)^{\star}=S^{\star}, \forall j$ ) the number of products per firm becomes the only individual state variable and we can write $v(n)$. All high productivity firms then solve

$$
v_{H}(n)=\max _{n^{\prime} \geq n\left(1-z^{\star}\right)}\left\{\pi_{H}\left(n, S^{\star}\right)-\left(n^{\prime}-n\left(1-z^{\star}\right)\right) \psi_{r}+\frac{1+g^{\star}}{1+r^{\star}} v_{H}\left(n^{\prime}\right)\right\} .
$$

[^1]Similarly, all low productivity firms solve

$$
v_{L}(n)=\max _{n^{\prime} \geq n\left(1-z^{\star}\right)}\left\{\pi_{L}\left(n, S^{\star}\right)-\left(n^{\prime}-n\left(1-z^{\star}\right)\right) \psi_{r}+\frac{1+g^{\star}}{1+r^{\star}} v_{L}\left(n^{\prime}\right)\right\} .
$$

The household's Euler equation yields $\frac{1+g^{\star}}{1+r^{\star}}=\beta\left(1+g^{\star}\right)^{1-\theta}$ and we have

$$
1+g^{\star}=\frac{Y_{t}}{Y_{t-1}}=\frac{Q_{t}}{Q_{t-1}}=\left[1+z^{\star}\left(\gamma^{\sigma-1}-1\right)\right]^{\frac{1}{\sigma-1}}
$$

The two accounting equations are again

$$
\begin{gather*}
S^{\star}=n_{H}^{\star} \phi J,  \tag{A17}\\
n_{H}^{\star} \phi J+n_{L}^{\star}(1-\phi) J=1 . \tag{A18}
\end{gather*}
$$

Finally, on the BGP we must have

$$
\begin{equation*}
n_{H}^{\star}=f_{H}\left(n_{H}^{\star}\right), \tag{A19}
\end{equation*}
$$

and

$$
\begin{equation*}
n_{L}^{\star}=f_{L}\left(n_{L}^{\star}\right), \tag{A20}
\end{equation*}
$$

where $f_{H}(\cdot)$ and $f_{L}(\cdot)$ are the policy functions of the high and low types. These equations fully characterize values of $S^{\star}, z^{\star}, n_{H}^{\star}$ and $n_{L}^{\star}$ on the BGP.

In the following we again focus on an interior BGP as defined in the main text. The interior BGP is characterized in the following proposition.

Proposition A2 An interior BGP features a $\left(n_{H}^{\star}, n_{L}^{\star}, S^{\star}, z^{\star}\right)$ combination that fulfills

$$
\begin{equation*}
\phi J n_{H}^{\star}=S^{\star} \text { and }(1-\phi) J n_{L}^{\star}+\phi J n_{H}^{\star}=1, \tag{A21}
\end{equation*}
$$

as well as the following research arbitrage equations for high and low
productivity firms respectively:

$$
\begin{align*}
\psi_{r} & =\frac{\widetilde{\pi}_{H}-\psi_{o} n_{H}^{\star}}{\beta^{-1}\left(1-z^{\star}+z^{\star} \gamma^{\sigma-1}\right)^{-\frac{1-\theta}{\sigma-1}}-1+z^{\star}}  \tag{A22}\\
\psi_{r} & =\frac{\widetilde{\pi}_{L}-\psi_{o} n_{L}^{\star}}{\beta^{-1}\left(1-z^{\star}+z^{\star} \gamma^{\sigma-1}\right)^{-\frac{1-\theta}{\sigma-1}}-1+z^{\star}} \tag{A23}
\end{align*}
$$

This is a system of four equations in four unknowns which can be solved.

## Derivation of expression for concentration and labor share

In this more general model with a CES production function there is now a difference between the fraction of lines provided by high productivity firm, $S^{*}$, and the sales weight of high productivity firm in the aggregate economy which we denote by $\widetilde{S}^{\star}$. Total sales of a firm of high type is along the BGP

$$
\int_{0}^{n_{H}^{\star}} p(i) y(i) d i=n_{H}^{\star} Y\left[S^{\star} \frac{\widetilde{P}_{t}^{\sigma-1}}{\left(\frac{\mu_{H H} w_{t}}{\varphi_{L} \Delta}\right)^{\sigma-1}}+\left(1-S^{\star}\right) \frac{\widetilde{P}_{t}^{\sigma-1}}{\left(\frac{\mu_{H L} w_{t}}{\varphi_{L} \Delta}\right)^{\sigma-1}}\right] .
$$

Sales of a firm of low type is given by

$$
\int_{0}^{n_{L}^{\star}} p(i) y(i) d i=n_{L}^{\star} Y\left[S^{\star} \frac{\widetilde{P}_{t}^{\sigma-1}}{\left(\frac{\mu_{L H} w_{t}}{\varphi_{L}}\right)^{\sigma-1}}+\left(1-S^{\star}\right) \frac{\widetilde{P}_{t}^{\sigma-1}}{\left(\frac{\mu_{L L} w_{t}}{\varphi_{L}}\right)^{\sigma-1}}\right]
$$

As a consequence, the sales share of high types in the total economy can be written as

$$
\widetilde{S}^{\star}=\frac{S^{\star}\left[S^{\star}\left(\frac{\mu_{H H}}{\Delta}\right)^{1-\sigma}+\left(1-S^{\star}\right)\left(\frac{\mu_{H L}}{\Delta}\right)^{1-\sigma}\right]}{S^{\star}\left[S^{\star}\left(\frac{\mu_{H H}}{\Delta}\right)^{1-\sigma}+\left(1-S^{\star}\right)\left(\frac{\mu_{H L}}{\Delta}\right)^{1-\sigma}\right]+\left(1-S^{\star}\right)\left[S^{\star} \mu_{L H}^{1-\sigma}+\left(1-S^{\star}\right) \mu_{L L}^{1-\sigma}\right]} .
$$

Finally, let us derive the expressions for the labor income shares with a CES
production function. The firm level labor share of a high type is given by

$$
\begin{equation*}
\lambda_{H}^{\star}=\frac{\int_{0}^{n_{H}^{\star}} w l(i) d i}{\int_{0}^{n_{H}^{\star}} p(i) y(i) d i}=\frac{S^{\star} \mu_{H H}^{-\sigma}+\left(1-S^{\star}\right) \mu_{H L}^{-\sigma}}{S^{\star} \mu_{H H}^{1-\sigma}+\left(1-S^{\star}\right) \mu_{H L}^{1-\sigma}} . \tag{A24}
\end{equation*}
$$

The firm level labor share for low-type is given by

$$
\begin{equation*}
\lambda_{L}^{\star}=\frac{\int_{0}^{n_{L}^{\star}} w l(i) d i}{\int_{0}^{n_{L}^{\star}} p(i) y(i) d i}=\frac{S^{\star} \mu_{L H}^{-\sigma}+\left(1-S^{\star}\right) \mu_{L L}^{-\sigma}}{S^{\star} \mu_{L H}^{1-\sigma}+\left(1-S^{\star}\right) \mu_{L L}^{1-\sigma}} . \tag{A25}
\end{equation*}
$$

The aggregate labor share is the sales-weighted average of the firm labor shares

$$
\begin{equation*}
\lambda^{\star}=\widetilde{S}^{\star} \lambda_{H}^{\star}+\left(1-\widetilde{S}^{\star}\right) \lambda_{L}^{\star} . \tag{A26}
\end{equation*}
$$

The within change in labor share is the unweighted average of the change in within firm labor share.

$$
\begin{equation*}
\frac{\phi\left(\lambda_{H, 1}^{\star}-\lambda_{H, 0}^{\star}\right)+(1-\phi)\left(\lambda_{L, 1}^{\star}-\lambda_{L, 0}^{\star}\right)}{\widetilde{S}_{0}^{\star} \lambda_{H, 0}^{\star}+\left(1-\widetilde{S}_{0}^{\star}\right) \lambda_{L, 0}^{\star}}, \tag{A27}
\end{equation*}
$$

where 0 denotes the initial and 1 the new BGP, respectively.

## C. 2 General distribution of process efficiency

In this extension, we depart from the assumption that there are only two types of firms with high vs. low level of process efficiency. Instead, we extend the framework to have $N$ types of firms, where $N$ can be any finite positive number. More precisely, let there be $N$ levels of process efficiency $\varphi_{1}>\varphi_{2}>\cdots>\varphi_{N}$ with the maximum efficiency gap $\frac{\varphi_{1}}{\varphi_{N}}<\gamma$. As in the baseline model, we assume that the number of firms and lines are large enough so that we can consider an equilibrium where the distribution of followers faced by each type is the same as the share of products by each type.

Let $\rho \equiv 1 / \beta-1$ and let us denote the markup by $\mu\left(j, j^{\prime}\right) \equiv \frac{\gamma \varphi_{j}}{\varphi_{j^{\prime}}}$. The BGP equilibrium consists of rate of creative destruction $z^{\star}$ and product share of each
type $\left\{s_{j}^{\star}\right\}_{j=1}^{N}$ that satisfy

$$
\begin{gathered}
\psi_{r}\left(\rho+z^{\star}\right)=1-\sum_{j^{\prime}=1}^{N} \frac{s_{j^{\prime}}^{\star}}{\mu\left(j, j^{\prime}\right)}-\frac{\psi_{o}}{J} \frac{s_{j}^{\star}}{\varphi_{j}}, \quad \forall j \in\{1 \cdots N\}, \\
\sum_{j=1}^{N} s_{j}^{\star}=1 \text { and } s_{j}^{\star}>0, \quad \forall j \in\{1 \cdots N\} .
\end{gathered}
$$

The first condition is just the first-order conditions for each type of firm while the second condition says that all types of firms are active and their product shares sum to 1 .

To solve for the equilibrium, let $s^{\star}$ denote the array of product shares $\left[s_{1}^{\star}, s_{2}^{\star}, \cdots, s_{N}^{\star}\right]^{\prime}, \Delta$ denote $\left[\Delta_{1}, \Delta_{2}, \cdots, 1\right]^{\prime}$ where $\Delta_{j} \equiv \varphi_{j} / \varphi_{N}$ and 1 denote $[1,1, \cdots, 1]^{\prime}$. We then define

$$
A=\left(\left(\begin{array}{c}
\frac{1}{\gamma \Delta_{1}} \boldsymbol{\Delta}^{\prime} \\
\frac{1}{\gamma \Delta_{2}} \boldsymbol{\Delta}^{\prime} \\
\vdots \\
\frac{1}{\gamma \Delta_{N}} \Delta^{\prime}
\end{array}\right)+\frac{\psi_{o}}{J}\left(\begin{array}{cccc}
\frac{1}{\varphi_{1}} & 0 & \cdots & 0 \\
0 & \frac{1}{\varphi_{2}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{\varphi_{N}}
\end{array}\right)\right)^{-1}
$$

With these notations, we can solve for the equilibrium quantities $\mathbf{s}^{\star}$ and $z^{\star}$ along an interior BGP. As long as $1^{\prime} A 1 \neq 0$, the solution to the equilibrium is given by

$$
\begin{align*}
& z^{\star}=\frac{\mathbf{1}^{\prime} A \mathbf{1}-1}{\psi_{r} \mathbf{1}^{\prime} A \mathbf{1}}-\rho, \quad z^{\star} \in(0,1)  \tag{A28}\\
& \mathbf{s}^{\star}=A \mathbf{1}\left[1-\psi_{r}\left(\rho+z^{\star}\right)\right]=\frac{A \mathbf{1}}{\mathbf{1}^{\prime} A \mathbf{1}}, s_{j}^{\star} \in(0,1) \tag{A29}
\end{align*}
$$

As in the baseline model with two types, $\mathrm{s}^{\star}$ does not depend on the R\&D cost $\psi_{r}$. The labor share of type $j$ and the aggregate labor share are respectively given by

$$
\mu_{j}^{-1}=\frac{\boldsymbol{\Delta}^{\prime} \mathbf{s}^{\star}}{\gamma \Delta_{j}}, \quad \mu^{-1}=\frac{\boldsymbol{\Delta}^{\prime} \mathbf{s}^{\star}}{\gamma}\left[\frac{1}{\Delta_{1}}, \frac{1}{\Delta_{2}}, \cdots, \frac{1}{\Delta_{N}}\right] \mathbf{s}^{\star} .
$$

## D Additional comparative statics at the calibrated BGP

## D. 1 Additional comparative statics

Figure A3: BGP markups and rate of creative destruction as $\psi_{o}$ changes


Note: The figure plots values for $S^{\star}, z^{\star}$ and $\mu^{\star}$ on the BGP when $\psi_{o}$ changes, holding fixed other parameters at the baseline values.

Figure A4: BGP process and allocative efficiency as $\psi_{o}$ changes


Note: The figure plots values for detrended labor productivity $Y /(Q L)$, aggregate process efficiency (PE), $\varphi_{L} \Delta^{S^{\star}}$, and allocative efficiency (AE) on the BGP as $\psi_{o}$ changes, holding fixed other parameters at the baseline values. AE is along the BGP given by $\left(\left(S^{\star}\right)^{2}+(1-\right.$ $\left.\left.S^{\star}\right)^{2}+S^{\star}\left(1-S^{\star}\right) \Delta^{-1}+S^{\star}\left(1-S^{\star}\right) \Delta\right)^{-1}$.

## D. 2 Diminishing returns in R\&D

In our calibration with changing overhead and $R \& D$ cost, a $10.5 \%$ increase in the R\&D cost parameter, $\psi_{r}$, accounts for about $40 \%$ of the decline in long-run growth. We can endogenize this increase in $\mathrm{R} \& \mathrm{D}$ costs as stemming from diminishing returns in research with respect to $n$. Suppose the cost of innovating on $x$ lines is given by $Y \psi_{r} n^{\nu} x / n$. On a BGP where $x / n=z^{\star}$ for all firms, aggregate $\mathrm{R} \& \mathrm{D}$ as a share of output can be written as $\widetilde{\psi}_{r} z^{\star}$ where $\tilde{\psi}_{r}=\psi_{r}\left[S^{\star} n_{H}^{\nu-1}+\left(1-S^{\star}\right) n_{L}^{\nu-1}\right]$. Higher $\widetilde{\psi}_{r}$ means lower aggregate R\&D efficiency. Our baseline model features $\nu=1$ and $\widetilde{\psi}_{r}=\psi_{r}$. When $\nu>1$, R\&D intensity increases with firm size and higher $S^{\star}$ endogenously raises $\widetilde{\psi}_{r}$ and lowers R\&D efficiency. For example, when $\nu=1.5$ as in De Ridder (2021), the observed rise in concentration from $45.9 \%$ to $58 \%$ raises $\widetilde{\psi}_{r}$ by $51 \%,{ }^{4}$ leading to a much bigger decline in growth. We did not go this route because it would make R\&D intensity increase markedly with firm size, contrary to available evidence. ${ }^{5}$

[^2]
## E Solving for the transition dynamics

This section lays out the numerical method used to compute the transition dynamics in Section 5 of the main text. Let $n_{t}$ be the number of product a firm holds and let $h_{t}$ be the share of these products where the firm faces a high productivity second-best producer. We use $m_{t} \equiv n_{t} h_{t}$ to denote the number of products a firm holds where the firm faces a high productivity second-best producer. The dynamic problem of a firm of type $j=H, L$ in the main text, can be expressed (after dividing the objective by $Q_{0}$ ) as

$$
\begin{align*}
& \max _{\left\{n_{s}, m_{s}\right\}_{s=1}^{\infty}}\left\{\pi_{j}\left(n_{0}, m_{0}\right)-\left(n_{1}-n_{0}\left(1-z_{1}\right)\right) \psi_{r}\right\} \frac{Y_{0}}{Q_{0}}  \tag{A30}\\
&+ \frac{\gamma^{z_{1}}}{1+r_{1}}\left\{\pi_{j}\left(n_{1}, m_{1}\right)-\left(n_{2}-n_{1}\left(1-z_{2}\right)\right) \psi_{r}\right\} \frac{Y_{1}}{Q_{1}} \\
&+ \frac{\gamma^{z_{1}}}{1+r_{1}} \frac{\gamma^{z_{2}}}{1+r_{2}}\left\{\pi_{j}\left(n_{2}, m_{2}\right)-\left(n_{3}-n_{2}\left(1-z_{3}\right)\right) \psi_{r}\right\} \frac{Y_{2}}{Q_{2}} \\
& \vdots \\
&+\prod_{\tau=1}^{t} \frac{\gamma^{z_{\tau}}}{1+r_{\tau}}\left\{\pi_{j}\left(n_{t}, m_{t}\right)-\left(n_{t+1}-n_{t}\left(1-z_{t+1}\right)\right) \psi_{r}\right\} \frac{Y_{t}}{Q_{t}}
\end{align*}
$$

for given $n_{0}$ and $m_{0} \equiv n_{0} h_{0}=n_{0} S_{0}$, and subject to

$$
\begin{gather*}
m_{t}=m_{t-1}\left(1-z_{t}\right)+S_{t-1}\left(n_{t}-\left(1-z_{t}\right) n_{t-1}\right), \quad t=1,2, \ldots  \tag{A31}\\
n_{t} \geq n_{t-1}\left(1-z_{t}\right), \quad t=1,2, \ldots \tag{A32}
\end{gather*}
$$

where

$$
\begin{equation*}
\pi_{H}\left(n_{t}, m_{t}\right)=m_{t}\left(1-\frac{1}{\gamma}\right)+\left(n_{t}-m_{t}\right)\left(1-\frac{1}{\Delta \gamma}\right)-\psi_{o} \frac{n_{t}^{2}}{2} \tag{A33}
\end{equation*}
$$

and

$$
\begin{equation*}
\pi_{L}\left(n_{t}, m_{t}\right)=m_{t}\left(1-\frac{\Delta}{\gamma}\right)+\left(n_{t}-m_{t}\right)\left(1-\frac{1}{\gamma}\right)-\psi_{o} \frac{n_{t}^{2}}{2} \tag{A34}
\end{equation*}
$$

The $j$ index only shows up through profit functions (A33) and (A34) because we start the transition dynamics from an initial BGP where $h_{0, j}=S_{0}$ for $j=H, L$. As a result, only the profit functions differ between the high and low type firms.

We can iterate (A31) backward to express $m_{t}$ as a function of past $n$

$$
\begin{equation*}
m_{t}=S_{t-1} n_{t}+\sum_{a=1}^{t-1}\left(S_{a-1}-S_{a}\right) n_{a} \prod_{b=1}^{t-a}\left(1-z_{a+b}\right) \quad \forall t=1,2, \ldots \tag{A35}
\end{equation*}
$$

We denote this function for $m_{t}$ by $m_{t}\left(\left\{n_{s}\right\}_{s=1}^{t}\right)$.
We can then derive the derivative of $m_{t+k}$ with respect to $n_{t}$ (we suppress the $j$ subscript since the expression is the same for the two types)

$$
\frac{\partial m_{t+k}\left(\left\{n_{s}\right\}_{s=1}^{t+k}\right)}{\partial n_{t}}=\left\{\begin{array}{cc}
0 & \text { if } k<0  \tag{A36}\\
S_{t-1} & \text { if } k=0 \\
\left(S_{t-1}-S_{t}\right) \prod_{b=1}^{k}\left(1-z_{t+b}\right) & \text { if } k>0
\end{array}\right.
$$

This is the effect of increasing the number of products in period $t$ by one unit on the number of products facing a high type second-best firm in period $t+k$ (while holding the number of product in all other periods constant). Adding a product in $t$ adds $\left(1-z_{t+1}\right)$ products in $t+1$. Product acquisition through innovation $x_{t+1}$ therefore needs to drop by $\left(1-z_{t+1}\right)$ to keep $n_{t+1}$ constant. All other $x_{\tau}$ for $\tau>t+2$ are then unchanged.

What is the effect on $m_{t+k}$ ? Adding a product in $t$ adds $S_{t-1}\left(1-z_{t+1}\right)$ products with a high-type follower in $t+1$ while lowering $x_{t+1}$ by $\left(1-z_{t+1}\right)$ in $t+1$ reduces high type follower by $S_{t}\left(1-z_{t+1}\right)$. The net effect on $m_{t+1}$ is $\left(S_{t-1}-S_{t}\right)\left(1-z_{t+1}\right)$. This change decays at the rate of creative destruction, which is captured by the $\prod_{b=1}^{k}\left(1-z_{t+b}\right)$ term. Hence what matters for $m_{t+k}, k>0$ is the change in the composition of the pool from which the additional product is drawn from, i.e., the difference between $S_{t}$ and $S_{t-1}$. If $S_{t}=S_{t-1}$, an increase in $n_{t}$ has no effect on $m_{t+k}$ for $k>0$. If $S_{t}>S_{t-1}$, the change reduces the number of products with high-type followers. Vice versa for $S_{t}<S_{t-1}$.

Substituting (A36) into (A33) and (A34) and taking derivatives with respect
to $n$ yields

$$
\frac{\partial \pi_{t+k, H}\left(n_{t+k}, m_{t+k}\left(\left\{n_{s}\right\}_{s=1}^{t+k}\right)\right)}{\partial n_{t}}=\left\{\begin{array}{cl}
0 & \text { if } k<0  \tag{A37}\\
\frac{1-\Delta}{\Delta \gamma} S_{t-1}+1-\frac{1}{\Delta \gamma}-\psi_{o} n_{t} & \text { if } k=0 \\
\frac{1-\Delta}{\Delta \gamma}\left(S_{t-1}-S_{t}\right) \prod_{b=1}^{k}\left(1-z_{t+b}\right) & \text { if } k>0
\end{array}\right.
$$

and

$$
\frac{\partial \pi_{t+k, L}\left(n_{t+k}, m_{t+k}\left(\left\{n_{s}\right\}_{s=1}^{t+k}\right)\right)}{\partial n_{t}}=\left\{\begin{array}{cc}
0 & \text { if } k<0  \tag{A38}\\
\frac{1-\Delta}{\gamma} S_{t-1}+1-\frac{1}{\gamma}-\psi_{o} n_{t} & \text { if } k=0 \\
\frac{1-\Delta}{\gamma}\left(S_{t-1}-S_{t}\right) \prod_{b=1}^{k}\left(1-z_{t+b}\right) & \text { if } k>0 .
\end{array}\right.
$$

It is useful to rewrite the objective function in (A30) before taking first-order conditions. First, we use the Euler equation for the household's problem to express the discount factors as

$$
\prod_{t=a}^{b} \frac{\gamma^{z_{t}}}{1+r_{t}}=\beta^{b-a+1} \frac{y_{a-1} c_{a-1}}{y_{b} c_{b}}
$$

where $y_{t} \equiv Y_{t} / Q_{t}$ and $c_{t}$ denotes consumption share of output $C_{t} / Y_{t}$. This consumption share can be expressed as

$$
\begin{align*}
c_{t} \equiv \frac{C_{t}}{Y_{t}} & =1-\frac{O_{t}}{Y_{t}}-\frac{Z_{t}}{Y_{t}}  \tag{A39}\\
& =1-\left(\phi n_{t H}^{2}+(1-\phi) n_{t L}^{2}\right) \frac{\psi_{o} J}{2}-\psi_{r} z_{t+1} \\
& =1-\left(\frac{S_{t}^{2}}{\phi}+\frac{\left(1-S_{t}\right)^{2}}{1-\phi}\right) \frac{\psi_{o}}{2 J}-\psi_{r} z_{t+1}=c\left(S_{t}, z_{t+1}\right)
\end{align*}
$$

Substituting this expression into the objective function (A30), dividing by $y_{0}$ and rearranging allows us to express the problem of a firm of type $j=H, L$ as

$$
\begin{align*}
\max _{\left\{n_{t j}\right\}_{t=1}^{\infty}} & \pi_{j}\left(n_{0 j}, m_{0 j}\right)+n_{0 j}\left(1-z_{1}\right) \psi_{r}  \tag{A40}\\
& +\sum_{t=1}^{\infty} \beta^{t} \frac{c_{0}}{c_{t}}\left\{\pi_{j}\left(n_{t j}, m_{t}\left(\left\{n_{s j}\right\}_{s=1}^{t}\right)\right)+\psi_{r} n_{t j}\left[\left(1-z_{t+1}\right)-\frac{c_{t}}{c_{t-1} \beta}\right]\right\}
\end{align*}
$$

subject to

$$
\begin{equation*}
n_{t j} \geq n_{t-1, j}\left(1-z_{t}\right), \quad t=1,2, \ldots \tag{A41}
\end{equation*}
$$

Using $\Lambda_{t j}$ to denote the Lagrangian multiplier on constraint (A41), the first-order conditions of the (A40) with respect to $n_{t j}, t>0$ are

$$
\begin{align*}
& \frac{\partial \pi_{j}\left(n_{t j}, m_{t}\left(\left\{n_{s j}\right\}_{s=1}^{t}\right)\right)}{\partial n_{t j}}+\Lambda_{t j}-\Lambda_{t+1, j}\left(1-z_{t+1}\right)  \tag{A42}\\
= & \psi_{r}\left[\frac{c_{t}}{c_{t-1} \beta}-\left(1-z_{t+1}\right)\right]+f_{j} \frac{1-\Delta}{\Delta \gamma}\left(S_{t}-S_{t-1}\right) \sum_{a=t+1}^{\infty} \beta^{a-t} \frac{c_{t}}{c_{a}} \prod_{b=1}^{a-t}\left(1-z_{t+b}\right)
\end{align*}
$$

and

$$
\Lambda_{t j} \geq 0, \quad n_{t j} \geq n_{t-1, j}\left(1-z_{t}\right), \quad \Lambda_{t j}\left(n_{t j}-n_{t-1, j}\left(1-z_{t}\right)\right)=0
$$

where $f_{j}=\Delta$ if $j=L$ and $f_{j}=1$ otherwise.
We will solve two such "representative" firm problem, one for the $H$ type and one for the $L$ type. First, we rewrite (A42) in an iterative format. Define

$$
\begin{equation*}
d_{t} \equiv \sum_{a=t+1}^{\infty} \beta^{a-t} \frac{c_{t}}{c_{a}} \prod_{b=1}^{a-t}\left(1-z_{t+b}\right) \tag{A43}
\end{equation*}
$$

One can show that

$$
\begin{equation*}
d_{t}=\beta\left(1-z_{t+1}\right) \frac{c_{t}}{c_{t+1}}\left(1+d_{t+1}\right) \tag{A44}
\end{equation*}
$$

Replacing $n_{t H}=\frac{S_{t}}{\phi J}$ and $n_{t L}=\frac{1-S_{t}}{(1-\phi) J}$ in (A37) and (A38) allows us to write

$$
\begin{equation*}
\frac{\partial \pi_{H}\left(n_{t H}, m_{t}\left(\left\{n_{s H}\right\}_{s=1}^{t}\right)\right)}{\partial n_{t H}}=S_{t-1} \frac{1-\Delta}{\gamma \Delta}+1-\frac{1}{\Delta \gamma}-\psi_{o} \frac{S_{t}}{\phi J} \equiv \frac{\partial \pi_{H}}{\partial n_{t H}}\left(S_{t-1}, S_{t}\right) \tag{A45}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial \pi_{L}\left(n_{t L}, m_{t}\left(\left\{n_{s L}\right\}_{s=1}^{t}\right)\right)}{\partial n_{t L}}=S_{t-1} \frac{1-\Delta}{\gamma}+1-\frac{1}{\gamma}-\psi_{o} \frac{1-S_{t}}{(1-\phi) J} \equiv \frac{\partial \pi_{L}}{\partial n_{t L}}\left(S_{t-1}, S_{t}\right) \tag{A46}
\end{equation*}
$$

Substituting (A44) to (A46) into (A42) yields the following set of equations for
each period $t>0$

$$
\begin{align*}
& \frac{\partial \pi_{H}}{\partial n_{t H}}\left(S_{t-1}, S_{t}\right)+\Lambda_{t H}-\Lambda_{t+1, H}\left(1-z_{t+1}\right)=\psi_{r}\left[\frac{c_{t}}{c_{t-1} \beta}-\left(1-z_{t+1}\right)\right] \\
&+\frac{(1-\Delta)\left(S_{t}-S_{t-1}\right) d_{t}}{\Delta \gamma},  \tag{A47}\\
& \frac{\partial \pi_{L}}{\partial n_{t L}}\left(S_{t-1}, S_{t}\right)+\Lambda_{t L}-\Lambda_{t+1, L}\left(1-z_{t+1}\right)=\psi_{r}\left[\frac{c_{t}}{c_{t-1} \beta}-\left(1-z_{t+1}\right)\right] \\
&+\frac{(1-\Delta)\left(S_{t}-S_{t-1}\right) d_{t}}{\gamma},  \tag{A48}\\
& d_{t}=d_{t-1} \frac{1}{\beta\left(1-z_{t}\right)} \frac{c_{t}}{c_{t-1}}-1,  \tag{A49}\\
& h_{t, H}=\left(h_{t-1, H}-S_{t-1}\right) \frac{S_{t-1}}{S_{t}\left(1-z_{t}\right)+S_{t-1},}  \tag{A50}\\
& h_{t, L}=\left(h_{t-1, L}-S_{t-1}\right) \frac{1-S_{t-1}}{1-S_{t}}\left(1-z_{t}\right)+S_{t-1}, \tag{A51}
\end{align*}
$$

and

$$
\begin{equation*}
\Lambda_{t j} \geq 0, n_{t j} \geq n_{t-1, j}\left(1-z_{t}\right), \Lambda_{t j}\left(n_{t j}-n_{t-1, j}\left(1-z_{t}\right)\right)=0, j=H, L . \tag{A52}
\end{equation*}
$$

Note that equations (A50) and (A51) use the equilibrium condition that $\frac{n_{t-1, H}}{n_{t H}}=$ $\frac{S_{t-1}}{S_{t}}$ and $\frac{n_{t-1, L}}{n_{t L}}=\frac{1-S_{t-1}}{1-S_{t}}$.

## Forward iteration algorithm

Given $\left(d_{t-1}, z_{t}, S_{t-1}, h_{t-1, H}, h_{t-1, L}\right)$ and the $\Lambda_{j} \mathrm{~s}$, equations (A47) to (A51) determine $\left(d_{t}, z_{t+1}, S_{t}, h_{t H}, h_{t L}\right)$. First, we guess that both types of firms have interior solution, i.e., $\Lambda_{t j}, \Lambda_{t+1, j}=0$ for $j=L, H$. Then, we can multiply (A47) by $\Delta$ and subtract (A48) to eliminate the $d_{t}$ term on the RHS. This yields

$$
\begin{equation*}
\Delta-1-\Delta \psi_{o} \frac{S_{t}}{\phi J}+\psi_{o} \frac{1-S_{t}}{(1-\phi) J}=(\Delta-1) \psi_{r}\left[\frac{c_{t}}{c_{t-1} \beta}-\left(1-z_{t+1}\right)\right] \tag{A53}
\end{equation*}
$$

Substituting in $c\left(S_{t}, z_{t+1}\right)$ from (A39) yields

$$
\begin{aligned}
& \frac{c_{t-1} \beta}{(\Delta-1) \psi_{r}}\left[\Delta-1-\Delta \psi_{o} \frac{S_{t}}{\phi J}+\psi_{o} \frac{1-S_{t}}{(1-\phi) J}\right] \\
= & {\left[1-\left(\frac{S_{t}^{2}}{\phi}+\frac{\left(1-S_{t}\right)^{2}}{1-\phi}\right) \frac{\psi_{o}}{2 J}-c_{t-1} \beta+\left(c_{t-1} \beta-\psi_{r}\right) z_{t+1}\right] } \\
z_{t+1} \equiv & \frac{\frac{c_{t-1} \beta}{(\Delta-1) \psi_{r}}\left[\Delta-1-\Delta \psi_{o} \frac{S_{t}}{\phi J}+\psi_{o} \frac{1-S_{t}}{(1-\phi) J}\right]-\left[1-\left(\frac{S_{t}^{2}}{\phi}+\frac{\left(1-S_{t}\right)^{2}}{1-\phi}\right) \frac{\psi_{o}}{2 J}-c_{t-1} \beta\right]}{c_{t-1} \beta-\psi_{r}} \\
\equiv & z\left(S_{t}, S_{t-1}, z_{t}\right)
\end{aligned}
$$

We can substitute $z\left(S_{t}, S_{t-1}, z_{t}\right)$ into (A47) to derive an equation with $S_{t}$ as the only unknown

$$
\begin{aligned}
\frac{\partial \pi_{H}}{\partial n_{t H}}\left(S_{t-1}, S_{t}\right) & =\frac{\Delta \frac{\partial \pi_{H}}{\partial n_{t H}}\left(S_{t-1}, S_{t}\right)-\frac{\partial \pi_{L}}{\partial n_{t L}}\left(S_{t-1}, S_{t}\right)}{\Delta-1} \\
& +\frac{1-\Delta}{\Delta \gamma}\left(S_{t}-S_{t-1}\right)\left(\frac{d_{t-1}}{\beta\left(1-z_{t}\right)} \frac{c\left(S_{t}, z\left(S_{t}, S_{t-1}, z_{t}\right)\right)}{c_{t-1}}-1\right)
\end{aligned}
$$

This is a quadratic function that solves for $S_{t}$. We choose the solution that is between 0 and 1 . When there are multiple solutions within 0 and 1 , we choose one that satisfy the interiority assumption and yields $z_{t}$ between 0 and 1.

We check the interiority condition in the following way. For the calibrated parameters, where the high type increase their market share, the $H$ types firms have higher returns to innovate that low type firms. Hence positive $z_{t}$ means the high type has positive $\mathrm{R} \& \mathrm{D}$. We only need to worry about the case when the low type does not innovate. For each period $t$, we first guess that the low type does innovate and solve for the low types innovation. If the innovation turns out to be less than zero, we set the value to zeros. This implies that $S_{t}=$ $1-\left(1-z_{t}\right)\left(1-S_{t-1}\right)$. Substituting this into (A47) yields
$S_{t-1} \frac{1-\Delta}{\gamma \Delta}+1-\frac{1}{\Delta \gamma}-\psi_{o} \frac{1-\left(1-z_{t}\right)\left(1-S_{t-1}\right)}{\phi J}=\psi_{r}\left[\frac{c_{t}}{c_{t-1} \beta}-\left(1-z_{t+1}\right)\right]+\frac{1-\Delta}{\Delta \gamma} z_{t}\left(1-S_{t-1}\right) d_{t}$.
This equation solves for $z_{t+1}$.
We initiate the algorithm with a guess for $\left(z_{1}, d_{0}\right)$ and set $S_{0}=h_{0, H}=h_{0, L}=$ $S_{\text {old }}^{\star}$. The algorithm has an outer loop and an inner loop. The inner loop holds
$d_{0}$ fixed and iterates on $z_{1}$. It iterates on (A47) to (A51) until $S_{t}$ is close to the new BGP $S_{\text {new }}^{\star}$. Then it uses bisection to update the guess of $z_{1}$. It increases $z_{1}$ if the last value of $z$ is lower than the new BGP $z_{\text {new }}^{\star}$ and reduce $z_{1}$ otherwise.

The inner loop yields a path of $\left(z_{t+1}, S_{t}\right)$ that converges to $\left(z_{\text {new }}^{\star}, S_{\text {new }}^{\star}\right)$ holding fixed $d_{0}$. However, the implied path of $d_{t}$ that may not converge to the BGP value of $d_{\text {new }}^{\star} \equiv \frac{\beta\left(1-z_{\text {new }}^{\star}\right)}{1-\beta\left(1-z_{\text {new }}^{\star}\right)}$. The outer loop uses bisection to update $d_{0}$ until $d_{t}$ also converges. It reduces $d_{0}$ if the inner loop overshoots and increases $d_{0}$ otherwise.

We stop the algorithm when $\left(d_{t}, z_{t+1}, S_{t}\right)$ approximately converges to the new balanced growth path. Suppose this happens after $T$ periods. Then we set $\left(d_{t}, z_{t+1}, S_{t}\right)$ for $t>T$ to their new BGP values and iterate forward until $\left(h_{t, H}, h_{t, L}\right)$ converges to the new BGP. We do not keep on iterating on $\left(d_{t}, z_{t+1}, S_{t}\right)$ until ( $h_{t, H}, h_{t, L}$ ) converges because (A49) is not stable outside of its fixed point. Since machine precision does not allow the algorithm to reach the exact fixed point, $d_{t}$ eventually explodes as we iterate forward.

## Effect of changes in $\psi_{r}$

Since $\psi_{r}$ affects both types of firms in the same way (see (A47) and (A48)), we guess that $S_{t}=S_{\text {old }}^{\star}$ for all $t>0$ and the economy jumps to the new balanced growth path in period 0 . Substituting this guess into the first-order conditions and using $\frac{\partial \pi_{H}}{\partial n_{t H}}\left(S_{\text {old }}^{\star}, S_{\text {old }}^{\star}\right)=\frac{\partial \pi_{L}}{\partial n_{t L}}\left(S_{\text {old }}^{\star}, S_{\text {old }}^{\star}\right)$, we can pin down $z_{t}, t \geq 1$ by

$$
\begin{aligned}
\psi_{r, \text { old }}\left[\frac{1}{\beta}-\left(1-z_{\text {old }}\right)\right] & =\psi_{r, \text { new }}\left[\frac{1}{\beta}-\left(1-z_{\text {new }}\right)\right], \\
c_{\text {new }} & =c_{\text {old }}+\left(\psi_{r, \text { new }}-\psi_{r, o l d}\right)\left[\frac{1}{\beta}-1\right], \\
d_{0} & =d_{\text {new }}^{\star}=\frac{\beta\left(1-z_{\text {new }}^{\star}\right)}{1-\beta\left(1-z_{\text {new }}^{\star}\right)}, \\
h_{t, H} & =S_{\text {old }}^{\star}, \quad h_{t, L}=S_{\text {old }}^{\star} .
\end{aligned}
$$

That is, an increase in innovation cost $\psi_{r}$ lowers the rate of creative destruction and increases the share of consumption.

## References

Autor, David, David Dorn, Lawrence F Katz, Christina Patterson, and John Van Reenen, "The fall of the labor share and the rise of superstar firms," Quarterly Journal of Economics, 2020, 135 (2), 645-709.

Cao, Dan, Erick Sager, Henry Hyatt, and Toshihiko Mukoyama, "Firm growth through new establishments," working paper, 2020.

Melitz, Marc J and Sašo Polanec, "Dynamic Olley-Pakes productivity decomposition with entry and exit," RAND Journal of Economics, 2015, 46 (2), 362-375.

Ridder, Maarten De, "Market power and innovation in the intangible economy," working paper, 2021.

Rinz, Kevin, "Labor market concentration, earnings, and inequality," Journal of Human Resources, 2022, 57 (S), S251-S283.

Rossi-Hansberg, Esteban, Pierre-Daniel Sarte, and Nicholas Trachter, "Diverging trends in national and local concentration," NBER Macroeconomics Annual, 2021, 35 (1), 115-150.


[^0]:    ${ }^{1}$ The rise in national concentration in Table A1 contrasts with falling local concentration documented by Rossi-Hansberg, Sarte and Trachter (2021) and ?. One explanation for the diverging trends is that the largest firms grew by adding establishments in new locations.
    ${ }^{2}$ Cao, Sager, Hyatt and Mukoyama (2020) document a similar pattern in the Quarterly Census of Employment and Wages data, and Rinz (2022) documents increasing number of markets with at least one establishment belonging to a top 5 firm.

[^1]:    ${ }^{3}$ Note that the $Q$ terms cancels out since the quality distribution in each H-L combination is identical to the aggregate $Q$.

[^2]:    ${ }^{4}$ We calculate the change as $\widetilde{\psi}_{r}^{1} / \widetilde{\psi}_{r}^{0}-1$ where $\widetilde{\psi}_{r}^{t}=\psi_{r}^{0}\left[S_{t}^{\star} n_{t H}^{\nu-1}+\left(1-S_{t}^{\star}\right) n_{t L}^{\nu-1}\right]$.
    ${ }^{5}$ According to the 2016 Business R\&D and Innovation Survey (BRDIS) Table 17, R\&D intensity of firms reporting R\&D declines with firm employment ( $3.5 \%$ for firms with 10 K or more employees vs. $5.2 \%$ for other firms). We combine BRDIS with Business Dynamics Statistics (BDS) to estimate the share of firms that report R\&D and find that unconditional R\&D intensity is $0.63 \%$ for $10 \mathrm{~K}+$ firms and $0.43 \%$ for the other firms. This translates to $\nu \approx 1$ because $10 \mathrm{~K}+$ firms are about 1600 times larger than the rest of the firms (2016 BDS). A caveat is that this evidence is mostly for firms in manufacturing.

